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Quantised De Sitter Space and the Connection to the Pauli Principle

W. ULMER†

Institute of Theoretical Physics and Chemistry, University of Stuttgart, 7 Stuttgart, Relenbergstr. 32 /BRD

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Abstract

It will be shown that the introduction of a fundamental length l_0 permits the definition of commutator rules between different observation systems, represented by the Poincaré groups. This fact leads to the model of a quantized De Sitter space, and the formulation of a non-local quantum field theory will be obtained. The Dirac spinors will be derived from the invariance of the quadratic form, defining De Sitter groups, and a connection to Pauli's exclusion principle can be understood by the same reason of a quantised space. A description of the structure of elementary particles involves a particular importance of the group $SU(3)$.

1. *Introduction*

The intention of this paper is to find a starting-point for an investigation of the connection between spin and statistics. Therefore, we look for a consequence by applying the time-definition of special relativity to a many-body-system, where the particles have to obey the uncertainty relation. It was Pauli (1940) who demonstrated by means of special relativity that particles with odd spin must be quantized in agreement with the exclusion principle, if the energy is to remain positive definite. For particles with integral spin, a similar postulate requires that the quantization proceeds according to Bose statistics. Pauli derived the fundamental result for particles without interactions. But some years later (Streater & Wightman 1964), the same result could be shown for interacting particles, if divergence problems do not exist. It is remarkable that only a relativistic theory can demonstrate this relationship between spin and statistics. This is the reason why this problem will be analysed with the help of the principles of special theory of relativity to understand the same connection, without using the above consistence postulate and having difficulties with interactions.

† Present address: Institute of High Energy Physics, University of Heidelberg, 69 Heidelberg, West Germany.

The following conventions of the special theory of relativity will be used: Greek indices run from 1 to 4, Latin indices from 1 to 3. We define the four-vectors x_μ' and x_μ, where $x_4^{(1)} = ict^{(1)}$, the speed vector $v_i = (v_1, v_2, v_3)$, where $v^2 = v_1{}^2 + v_2{}^2 + v_3{}^2$, and

$$L^2 = x^\mu x_\mu = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 \tag{1.1}$$

$$\Box = \partial^2/\partial x^\mu \partial x_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{1.2}$$

A homogeneous proper Lorentz transformation has to satisfy

$$x_\mu{}^1 = L^{\mu\nu} x_\nu, \qquad x^\mu x_\mu = (x^\mu)'(x_\mu)' \tag{1.3}$$

$$L_\mu{}^\nu L_\nu{}^\rho = \delta_\mu{}^\rho, \qquad \det |L| = +1 \tag{1.4}$$

As we have to consider first Lorentz transformations, where $L_j{}^i$ must have the following form

$$L_j{}^i = ((1 - v^2/c^2)^{-1/2} - 1)\frac{v_i v_j}{v^2} + \delta_j{}^i$$
$$L_4{}^4 = (1 - v^2/c^2)^{-1/2} \tag{1.5}$$

we do not discuss here transformations with $\det |L| = -1$. It is important to extend (1.3) to the inhomogeneous Lorentz transformation by adding a constant four-vector a_μ to equation (1.3):

$$x_\mu' = L_\mu{}^\nu x_\nu + a_\mu \tag{1.6}$$

This equation enables a discussion of the translation group. For detailed representations of the Lorentz group, one should consult a recent work by Streater and Wightman (1964).

The introduction of the inhomogeneous Lorentz group involves the use of the homogeneous quadratic form

$$-S_\mu{}^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 - c^2 t^2 - a_\mu{}^+ a_\mu \tag{1.7}$$

which may be comprehended as a De Sitter group for each μ.

2. *Relativistic Treatment of a Many-Body-System*

For every non-relativistic theory it is characteristic that the investigation of a many-body-system with a single time for the whole problem is sufficient. If a system of interacting particles with static interactions is considered, which must not be assumed in special relativity, the satisfaction of the Pauli principle by antisymmetrical wave functions leads to the 'exchange-interactions' of quantum mechanics. The fundamental principle of special relativity—the equivalence of each observation system and the finite speed of interactions or informations, transmitted by a light beam or a field—applies to a many-particle-system in the same way as for a many-time-system (Dirac, 1935; Landau & Lifschitz, 1967). Because we may consider electrons or positrons as 'particles' with a surrounded 'photoncloud'—and, using this analogy, the nucleons as 'particles' which are surrounded by a 'meson-cloud'—we should try to classify the particles according to the above consideration about the finite propagation speed of items of information. A classification of particles according to the spin will result by use of a mathematical formulation.

(1) Particles of which the state is an object of physical informations (space coordinates, speed, momentum, energy, and so on). The uncertainty relation does not allow the measurement of all the above information at the same time.
(2) Particles which are used for the transmission of interactions or items of information (photons to obtain space coordinates of electrons).

In this picture, the Coulomb interaction between two charged particles is represented by an exchange of photons. The emission or absorption of a photon by an electron can be characterised through the fact that a particle represents a 'source' or a 'dip' of physical information. Another point seems to be important. Heisenberg (1942) emphasised in one of his papers that the description of particles by a probability behaviour of a 'point charge' cannot be justified in very short distances of space and time. The 'overlapping' of 'photon-clouds' is in this case very strong, and therefore we have to take into account the creation or annihilation of new particles. Because of these problems in high-energy physics we wish to start with 'gedanken experiments', which should lead to a key for understanding the exclusion principle in a qualitative way, using from the outset the principles of special relativity. Two particles and a measurement apparatus form, in the special theory of relativity, three equivalent partial systems. Let us consider first the two particles. Each one of them can be regarded as motionless.

Particle 1 rests in $\sum' (x_\mu{}^1)'$, and with respect to this system particle 2 shall have the coordinates $\sum (x_\mu{}^2)$. Particle 2 rests in $\sum' (x_\mu{}^2)'$, and the coordinates of particle 1 are given by $\sum (x_\mu{}^1)$. If we remember the famous question of Einstein: 'What happens at the same time?', then we must conclude that this statement has no physical meaning without any measurement. Therefore, the emission of a light signal is needed in order to obtain the necessary information for a determination of $\sum (x_\mu{}^2)$, if the observation point was specified to $\sum' (x^1)'$. According to the principles of special relativity an exact measurement of x_μ and v_i is always possible. In particular, a synchronisation of t_1' and t_2 can be performed, if there is no uncertainty relation which restricts the information by a measurement.

It is absolutely necessary to emit light signals in order to obtain those items of information which are required for an application of the Lorentz transformation (1.6)

$$L_\mu{}^\nu x_\nu + a_\mu = x_\mu{}'$$

and to determine the whole system. In quantum theory, it will be shown that a light beam has to satisfy the uncertainty relation $\Delta x . \Delta p_x \geqslant \hbar/2$, It is interesting to note that Heisenberg's uncertainty relation for a material particle (for example, an electron), can be formulated only when the light beam also satisfies the uncertainty relation. If there were no restriction on Δx and Δp_x for the light beam, we would be able to determine the position of the electron to an infinite degree of accuracy by a very well-localised beam without transferring an appreciable momentum uncertainty to the electron in an uncontrollable way (Sakurai, 1967). Then we could also determine the electron by virtue of transformation (1.6). In quantum theory, the state of two particles is described by $\phi(t_1)$ and $\phi(t_2)$. Hence $\phi(t_2)$ and t_1' cannot be obtained by one measurement, as there is no possibility of a synchronisation of t_1' and t_2. But a formulation of a commutator rule is not possible. A consequence of the impossibility to synchronise t_1' and t_2, is the following agreement with the exclusion principle. For particles of the first kind of our classification we cannot find a condition which enables a proof of the statement that more than one particle occupies a quantum state at the same time, because this statement can never be the result of a measurement.

Now we should investigate another system. We take one particle of the first kind of our classification, which shall rest in $\sum' (x_\mu)'$. By means of a photon-scattering, a measurement apparatus will observe this particle and fix its coordinates very sharply to $\sum^0 (x_\mu)$. The momentum or the relative speed are not defined. The application of the Lorentz transformation (1.6) leads our considerations to the same problem as discussed before. The uncertainty relation enables either the knowledge of x_μ or v_i. The knowledge of both items of information is not possible, as we need an interacting field. Because a light beam, described by the Maxwell equations, which remains invariable under a Lorentz transformation has to obey, in addition to the uncertainty relation, we may conclude that the determination of x_μ' and x_μ by means of the Lorentz transformation cannot be taken into account. This is also valid for any other interaction fields, not only for the electromagnetic field, as, according to the principles of relativity, the Lorentz transformation must be the fundament of every field equation. If we wish to maintain the Lorentz transformation for the description of the connection between x_μ' and x_μ, then this transformation must contain statistical information. From a relativistic starting-point, where we must not prefer the time coordinate, we may require for microparticles a non-vanishing commutator

$$x_\mu x_\nu' - x_\nu' x_\mu \neq 0 \tag{2.1}$$

Because of the impossibility to determine all information of an application of a Lorentz transformation, this commutator must exist. Therefore, we have to answer the question, whether it is necessary to start with such a commutator relation. In the usual relativistic quantum theory we require only Lorentz invariance of a measurement, which has been performed by any two classical apparatuses. If the first measurement apparatus observes $\Psi(\Sigma)$, then the other apparatus must observe $\tilde{\Psi}(\tilde{\Sigma}) = u.\Psi(\Sigma)$, where u is an unitary transformation. But we wish not only to regard a particle from different observation systems which are moved against each other but also to define those coordinates where a particle rests, and only in that system is it justified to use a rest mass of a particle m_0 for any calculation. The above commutator will lead to the model a of non-local quantum field theory. The connection between the exclusion principle and the relativistic many-particle-problem is given by the non-existence of a single time for the state vectors. Therefore, the determination of a many-time-problem with the help of the Lorentz transformations is not possible, because the uncertainty relation does not allow us to know all the information required for the application of the Lorentz transformation. The introduction of commutator rules of the kind $\Psi(t)t' - t'\Psi(t) \neq 0$ or $\Psi(t)\Psi'(t') - \Psi'(t')\Psi(t) \neq 0$ seems to be justified in a qualitative way, but the definition of commutator rules is very difficult to formulate. It is easy to verify that every function $\phi(x_\nu)$ cannot commute with any other function $\phi'(x_\mu')$, if the arguments x_ν do not commute with x_μ'.

Now we wish to investigate the application of relation (2.1) to the quadratic form (1.7) of the De Sitter groups:

$$-S_\mu{}^2 = x^2 + y^2 + z^2 - c^2 t^2 - a_\mu{}^+ a_\mu$$

This relation (2.1) must be invariable under a homogeneous, proper Lorentz transformation, and therefore it is necessary to postulate the commutator

$$\begin{aligned} x_\nu x_\mu' - x_\mu' x_\nu &= l_0{}^2 \gamma_{\mu\nu} \\ x^\nu x^{\mu'} - x^{\mu'} x^\nu &= l_0{}^2 \gamma^{\mu\nu} \\ x^\nu \gamma_\mu x^{\mu'} - \gamma_\mu x^{\mu'} x^\nu &= l_0{}^2 \gamma_\mu \gamma^{\mu\nu} \\ &\equiv l_0{}^2 \gamma^\nu \end{aligned} \tag{2.2}$$

The Lorentz invariance of (2.2) is guaranteed, if $\gamma_{\nu\mu}$ satisfies the relation

$$\tilde{\gamma}_{\lambda\mu} = u_\lambda{}^\nu \gamma_{\nu\mu}.$$

We should emphasize that other properties of $\gamma_{\nu\mu}$ are *only* defined in connection with the quadratic form (1.7). A relation between space coordinates and speed vector can be found by means of the Lorentz transformation, which depends on v_i. This will be our problem. A speed greater than that of light is impossible. To find a field equation, we shall use a method very well-known from mathematics. In quantum theory each canonical quantisation

$$[q_i, p_j] = i\hbar\delta_{ij}$$

may be transformed by virtue of

$$\left[\frac{\partial}{\partial q_i} q_i - q_i \frac{\partial}{\partial q_i}\right] F(q_i) = F(q_i)$$

in special representations of the dynamical variables. We are able to get the Schrödinger equation as well as the relativistic Klein–Gordon equation.

In the same way we can represent equation (2.2), using the above relation for differential operators, if we form linear combinations

$$aSx_1'(x_\nu)\,\phi = -l_0^2\,\gamma_{11}\frac{\partial\phi}{\partial x_1} = \cdots = -l_0^2\,\gamma_{14}\frac{\partial\phi}{\partial x_4}\,.$$

Each operator x_μ' can be represented by a linear combination of four differential operators. For this purpose we use the symbol γ^ν, which denotes that we form four linear combinations of the introduced matrix $\gamma_{\nu\mu}$. Now we will obtain the equation

$$\gamma^\mu x_\mu'(x_\nu)\,\phi = -l_0^2\,\gamma^\nu\frac{\partial\phi}{\partial x_\nu}$$

$$\gamma^\mu(L_\mu{}^\nu x_\nu + a_\mu)\,\phi = -l_0^2\,\gamma^\nu\frac{\partial\phi}{\partial x_\nu} \tag{2.3}$$

It is not difficult to verify that the introduced l_0 must have the dimension of a length. This equation has to satisfy the principles of relativity, which are guaranteed, if the introduced matrices γ^ν agree with Dirac spinors

$$\gamma^\nu\gamma^\lambda + \gamma^\lambda\gamma^\nu = 2\delta^{\nu\lambda} \tag{2.4}$$

The reason for this fact is the quadratic form (1.7). In addition, we obtain the relation

$$a^\nu\gamma^\lambda + \gamma^\lambda a^{\nu+} = 0 \tag{2.5}$$

The invariance of (2.3) under a translation group is guaranteed. If we replace $x_\nu \to x_\nu + a_\nu'$, then we shall get

$$a_\nu' x_\nu + x_\nu a_\nu'^{+} = 0 \tag{2.6}$$

Therefore, we observe that a_μ is satisfied by the conditions

$$ia^4 = (ia_4)^+ = -ia_4, \qquad a^i = a_i^+ = -a_i, \qquad a_i^+ a_i = -a_i^2 \quad (i = 1, 3),$$
$$\mu = 4 \;\Rightarrow\; (ia_4)(ia_4)^+ = a_4^2$$

With the help of the quadratic form (1.7) we obtain a second-order equation

$$(x^2 + y^2 + z^2 - c^2t^2 - a_\mu^+ a_\mu)\,\phi_\mu = l_0^4\,\Box\phi_\mu \tag{2.7}$$

The special case $a_\mu = 0$ automatically satisfies relation (2.6). There exists some interesting special conditions of equations (2.3) and (2.5). First we choose the corresponding physical conditions, then we are able to get two equations, which are very well known from relativistic quantum theory for instance, if we regard a particle from very far distances

$$\gamma^\nu l_0^2\frac{\partial\phi}{\partial x_\nu} = -a_\mu\,\phi$$

$$l_0^4\,\Box\phi = a_\mu^+ a_\mu\,\phi \tag{2.8}$$

For this reason it is justifiable to suppose that the constant $a\mu$ has something to do with the rest mass of a particle. A specification to the homogeneous

Lorentz group, where $a_\mu \equiv 0$, leads directly to the Weyl equation, represented by Dirac spinors

$$\gamma^\nu \frac{\partial \phi}{\partial x_\nu} = 0 \tag{2.9}$$

Equation (2.3) forms the basis of a discussion for particles with odd spin which have to obey to the Pauli principle; equation (2.7) must be used for a discussion of Bose particles. In Section 3 we will show that we obtain by means of equations (2.3) and (2.7) a quantised De Sitter space, where the curvature can only accept discrete values. The group SU(3) seems to play a very important role, because the absence of strong interactions will always lead to the condition $a_1 = a_2 = a_3$, or: $a_1' = a_2' = a_3'$. Therefore, we can hope to calculate the rest mass of elementary particles when more information about l_0 is known.

3. *Solution of the Field Equations*

Equation (2.7) can be solved by the introduction of creation and annihilation operators. We separate $\phi_\mu(x_\nu) = R_\mu(x_i) T_\mu(t)$ by means of a constant λ^2 on both sides

$$\frac{c^2 t^2}{2l_0^2} T_\mu(t) - \frac{l_0^2}{2c^2} \frac{\partial^2 T_\mu(t)}{\partial t^2} = \frac{\lambda^2 + a_\mu^2}{2l_0^2} T_\mu^t$$

$$\frac{x^i x_i}{2l_0^2} R_\mu(x_i) - \frac{l_0^2}{2} \Delta R_\mu(x_i) = \frac{\lambda^2}{2l_0^2} R_\mu(x_i)$$

Like the harmonic oscillator in quantum theory we define operators, which have to obey the commutator rules for Bose particles

$$b_t^+ = \frac{1}{\sqrt{2}}\left(\frac{c}{l_0} t - \frac{l_0}{c}\frac{\partial}{\partial t}\right), \qquad b_t = \frac{1}{\sqrt{2}}\left(\frac{c}{l_0} t + \frac{l_0}{c}\frac{\partial}{\partial t}\right),$$

$$b_i^+ = \frac{1}{\sqrt{2}}\left(\frac{x_i}{l_0} - l_0 \frac{\partial}{\partial x_i}\right), \qquad b_i = \frac{1}{\sqrt{2}}\left(\frac{x_i}{l_0} + l_0 \frac{\partial}{\partial x_i}\right),$$

$$[b_k, b_l^+] = \delta_{kl}, \qquad [b_t, b_t^+] = 1$$

$$[b_k^+, b_l^+] = [b_k, b_l] = [b_t^+, b_t^+] = 0$$

$$[b_t, b_t] = [b_k, b_t] = [b_k^+, b_t^+] = 0$$

By that means we obtain the results

$$\lambda^2 = 2l_0^2 \left(\sum_{i=1}^{3} b_i^+ b_i + \tfrac{3}{2} \right) = 3l_0^2 + M . 2l_0^2$$

$$M = 2m + l = 0, 1, 2, \ldots \qquad (m, l = 0, 1, \ldots,) \tag{3.1}$$

$$b_t^t b_t T_\mu = n T_\mu, \qquad n + \tfrac{1}{2} = (\lambda^2 + a_\mu^2)/2l_0^2 \qquad (n = 0, 1, \ldots) \tag{3.2}$$

$$\lambda^2 \gtreqless a_\mu^2, \quad a_\mu^2 = 2l_0^2(n - M - 1) \qquad (n \geqslant M + 1; \quad \mu = 1, \ldots 3) \tag{3.3}$$

$$M = 2m + l, \text{ resp. } l \leqslant m - 1$$

$$a_4^2 = 2l_0^2(M + 1 - n) \qquad (n \leqslant M + 1)$$

The conditions $a_\mu \equiv 0$ resp. $a^i a_i = a_4{}^2$, which satisfy the anticommutation rules (2.5), are able to be realised by (3.3). It is worth mentioning that the harmonic oscillator (3.1)

$$\lambda^2 = 2l_0{}^2\left(\sum_{i=1}^{3} b_i{}^+ b_i + \tfrac{3}{2}\right)$$

and its degeneration degree is discussed in detail by Messiah (1970).

The solution of the four components $\phi_\mu(x_\nu)$ is given by

$$\phi_\mu(x_\nu) = N.\exp\{-l_0^{-2}.c_\mu{}^\nu x_\nu x_\nu\}.\exp(\pm ik_\nu x_\nu) \qquad (3.4)$$

We substitute (3.4) into (2.3), and then we will get

$$L_\mu{}^\nu = 2\gamma_\mu{}^\nu.c_\mu{}^\nu$$

$$a_\mu = \mp il_0{}^2 k_\nu \gamma_\mu{}^\nu \qquad \left(k_4 = \frac{\omega}{c} = \frac{\overbrace{mc^2}^{E^0}}{\hbar c}\right) \qquad (3.5)$$

It is not difficult to see that the solution of the one-particle-equation is given by a very complicated Gaussian function and plane waves. The Dirac equation of a free particle has only the latter solution, and corresponds to equation (2.3) if a particle is observed from infinity. But the very rapid descension of Gaussian functions permits us to regard a particle in a distance of several lengths l_0 as a 'point'.

In the special theory of relativity there exists a profound relationship between matter and energy. Using the equivalence between matter and field we are able to describe the quantisation of the energy-momentum-tensor by means of equation (2.3):

$$\gamma^\mu(L_\mu{}^\nu x_\nu + a_\mu)\,\phi = -l_0{}^2\gamma^\nu\frac{\partial\phi}{\partial x_\nu}$$

We will do this here in the case of the electromagnetic field. It is not necessary to couple a particle to an external field, because we can easily verify a possible way to obtain expressions for the total energy, momentum, and angular momentum

$$E = mc^2(1 - v^2/c^2)^{-1/2} = mc^2 L_4{}^4$$

The first three components of the relativistic momentum are obtained by $mcL_i{}^4$ resp. $mcL_4{}^i$, and doing the same with the remaining components we shall get the corresponding expression for the relativistic angular momentum. But we wish to compare equation (2.3) with the energy-momentum-tensor of the electromagnetic field

$$T_{ij} = \frac{1}{4\pi}(E_i E_j + H_i H_j) - \tfrac{1}{2}(E^l E_l + H^l H_l))\,\delta_{ij}$$

$$T_{44} = \frac{1}{\gamma\pi}(E^l E_l + H^l H_l)$$

The electromagnetic mass is given by U_4/c^2, the total energy by

$$U_4 = mc^2 L_4{}^4 = \int T_{44}\,dT$$

In order to find a connection to $T_{\mu\nu}$ and to guarantee the satisfaction of the conservation of total energy, we have to introduce a 'structure-tensor' for the electromagnetic rest mass of a charged particle: $m_{\mu\nu}$.

In addition, there must exist a connection between the symmetry of $m_{\mu\nu}$ and $T_{\mu\nu}$, as from the conservation of total energy that follows:

$$\mathrm{Tr}\, m_{ii} = -m_{44} = -m$$

If the electromagnetic field shows invariance under symmetry transformations of the rotation group, then we can conclude that

$$m_{11} = m_{22} = m_{33} = -m/3$$

But we should mention here that a scattering problem of three or more particles, where the 'photon-clouds' show a strong 'overlapping', does not allow this conclusion, because a more complicated tensor $m_{\mu\nu}$ is needed.

Appendix

We shall now look for a possible way to discuss a two-particle-equation. A generalisation to a many-particle-field equation is then not difficult. As two particles are observed from a measurement apparatus Σ^0, we introduce an operator $P_0{}^1$, which shows that particle 1 is regarded by Σ^0. Then we shall obtain both the Lorentz transformations

$$(x_\mu{}^1)' = P_0{}^1(L_\mu{}^\nu x_\nu + a_\mu)$$
$$(x_\mu{}^2)' = P_0{}^2(L_\mu{}^\nu x_\nu + a_\mu) \tag{A.1}$$

The corresponding field equations are given by

$$\gamma^{1\mu}(x_\mu{}^1)'\,\phi = -P_0{}^1\left(l_0{}^2\gamma^\nu \frac{\partial \phi}{\partial x_\nu}\right)$$
$$\gamma^{2\mu}(x_\mu{}^2)'\,\phi = -P_0{}^2\left(l_0{}^2\gamma^\nu \frac{\partial \phi}{\partial x_\nu}\right) \tag{A.2}$$

Our starting-point was the fact that we can never obtain any information about particles without interacting by means of a field. Therefore, the corresponding uncertainty principle should contain the field equation, and the commutator rules must represent a particle as a source of an interacting field. According to the principles of special relativity, we can observe the other particle from the corresponding rest system of one particle. The introduction of a commutator rules between the particles describes the interacting field.

$$(x_\mu{}^1)' = P_2{}^1(L_\mu{}^\nu x_\nu + a_\mu) \;\Rightarrow\; \gamma^\mu(x_\mu{}^1)'\,\phi = -P_2{}^1\left(l_0{}^2\gamma^\nu \frac{\partial \phi_\mu}{\partial x_\nu}\right)$$
$$(x_\mu{}^2)' = P_1{}^2(L_\mu{}^\nu x_\nu + a_\mu) \;\Rightarrow\; \gamma^\mu(x_\mu{}^2)'\,\phi = -P_1{}^2\left(l_0{}^2\gamma^\nu \frac{\partial \phi_\mu}{\partial x_\nu}\right) \tag{A.3}$$

It is to the aim to formulate a product space with respect to Σ^0. The difficulty is to define a scalar product, as the introduced ϕ is only a Hilbert space for far distances; however, we are reminded of quantum mechanics, where a product space of interacting particles $H(1,2)$ involves a non-linear, self-consistent, field equation. Therefore, we will look for the possibility of finding a connection to the very well-known non-linear field equations in quantum field theory.

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